Spectral characteristics of dual header pulse interval modulation (DH-PIM)

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Abstract: The authors present a detailed derivation of the power spectral density of the DH-PIM scheme. It is shown that the spectral profile resembles a general sinc envelope shape, which contains DC and potential distinct slot components. The amplitude of both components and the location of the slot component largely depend on the bit resolution and the pulse duty cycle. To confirm the validity of the mathematical model, a computer simulation model is also developed and the results presented are in close agreement with the predicted results.

1 Introduction

There are many digital modulation techniques, which have been suggested for use in optical fibre communications and optical wireless communication systems [1-4]. Digital pulse time modulation techniques are well known schemes, which utilise the vast optical bandwidth to provide improved performance with less circuit complexity compared with pulse code modulation and on-off-keying (OOK) schemes [5]. Out of many such schemes, pulse position modulation (PPM) achieves the best performance at a very low average power [2, 3], whereas pulse interval modulation (PIM) offers marginally lower performance but improved transmission capacity and a built-in symbol synchronisation capability compared with PPM [6, 7].

However, with an emerging Internet technology there is a need for a new digital modulation scheme which offers even higher transmission capacity than both PPM and PIM and delivers the same performance as that of PIM. Dual header pulse interval modulation (DH-PIM) is the modified version of PIM, which was first proposed in 1999. It offers higher transmission capacity and better bandwidth efficiency, but at the expense of higher power requirements compared with PPM and PIM [8, 9]. Many researchers [7, 10, 11] have considered the evaluation of the PIM and PPM spectra. However, no work has been reported on the spectral characteristics of DH-PIM. In this paper we introduce the DH-PIM scheme and present an original study of its spectral characteristics including expressions for the pulse train and power spectral density. The mathematical model developed is evaluated and the results obtained are verified with the simulation data.

2 System theory

Simulation of and difference between the OOK, PPM, PIM and DH-PIM formats may be best understood by reference to Fig. 1, in which an example of two consecutive symbols (frames) is shown. In PPM, a pulse of one time slot duration is located in the one slot corresponding to the information data. In PIM the symbol length is shortened by eliminating redundant void-slots which follow the PPM pulse, thus resulting in an increased data throughput [7, 10]. DH-PIM reduces the symbol length further, compared with PIM, by assigning one of the two possible headers at the start of each symbol in order to achieve even higher data throughput as outlined below.

![Fig.1 OOK/PPM/PIM/DH-PIM symbol structure using α = 1 and α = 2 as examples for DH-PIM](image)

In DH-PIM [8, 9], each M-bit OOK input sequence is encoded into a symbol starting with a header of (α + 1)T_s duration followed by dT_s empty slots corresponding to the information. Here, T_s is the slot duration, α > 0 is a positive integer and d is a variable which represents the information. There are two types of headers, both start with a pulse and end with a space. In header one, the pulse has a duration of αT_s/2 while in header two the duration is αT_s. To avoid symbols, which have no slots between adjacent pulses, the header also includes a trailing guard band of suitable duration (T_g) as shown in Figs. 1 and 2. If the most significant bit (MSB) of the OOK binary word is zero, the DH-PIM symbol starts with header one followed...
by a number of empty time slots equal to the decimal value of the binary input word. If MSB = 1, the symbol starts with header two followed by a number of empty time slots equal to the decimal value of the 1's complement of the binary input word. A pulse plays a dual role of symbol initiation and time reference for the preceding and succeeding symbols.

DH-PIM not only removes the redundant time slots that follow the pulse as in PPM symbol, but it also, for α = 1, reduces the average symbol length to half that of PIM and quarter that of PPM as shown in Fig. 1.

The spectral properties of DH-PIM are investigated in the next Section.

3 Spectral properties

Fig. 2 shows a typical DH-PIM structure for the nth symbol. It can be represented as a rectangular pulse that starts at t = Tn and has a duration of τ = (1 + hₙ)αT/2, where hₙ ∈ {0, 1} indicating header 1 or header 2, respectively, and n is the instantaneous symbol number. DH-PIM pulse train can be expressed mathematically as:

\[ x(t) = V \sum_{n=0} \text{rect} \left[ \frac{2(t - T_n)}{\alpha T} - 1 \right] \]

\[ + hₙ \text{rect} \left[ \frac{2(t - T_n)}{\alpha T} - 3/2 \right] \]

where V is the pulse amplitude, and the rectangular pulse function is defined as [12]:

\[ \text{rect}(u) = \begin{cases} 1 & -0.5 < u < 0.5 \\ 0 & \text{otherwise} \end{cases} \]

For hₙ = 0, eqn. 1 represents DH-PIM symbols with header one only having pulse duration of τ = αT/2.

The start of the nth symbol is defined as:

\[ T_n = T_0 + T_s \left( n(\alpha + 1) + \sum_{k=0}^{n-1} d_k \right) \]

where T₀ is the start time of the first pulse at n = 0 and dᵦ ∈ {0, 1, ..., 2ⁿ⁻¹ − 1} represents the number of information time slots in the nth symbol.

The Fourier transform [13] of the truncated signal xₙ(t) of N symbols is given as:

\[ X_N(\omega) = V \sum_{n=1}^{N-1} \int_{-\infty}^{\infty} \text{rect} \left[ \frac{2(t - T_n)}{\alpha T} - 1 \right] e^{-j\omega t} dt \]

\[ + hₙ \text{rect} \left[ \frac{2(t - T_n)}{\alpha T} - 3/2 \right] e^{-j\omega t} dt \]

\[ X_N(\omega) = V \sum_{n=0}^{N-1} \left\{ \int_{T_n}^{T_n + \frac{T_s}{2}} e^{-j\omega t} dt \right\} \]

\[ + hₙ \int_{T_n + \frac{T_s}{2}}^{T_n + \frac{T_s + T_s}{2}} e^{-j\omega t} dt \]

\[ X_N(\omega) = \frac{V}{j\omega} e^{-j\omega T_0} \left( 1 - e^{-j\omega T_s} \right) \]

\[ \times \sum_{n=0}^{N-1} \left[ 1 + hₙ e^{-j\omega T_s} e^{-j\omega T_s(n+1)} \sum_{k=0}^{n-1} d_k \right] \]

\[ \times e^{-j\omega T_s(n+1)(n+2)/2} \]

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\[ \times \sum_{n=0}^{N-1} \left[ 1 + hₙ e^{-j\omega T_s} e^{-j\omega T_s(n+1)} \sum_{k=0}^{n-1} d_k \right] \]

The expression in eqn. 9 is best evaluated by splitting it into three regions, (i) S₁(ω) where q < n, (ii) S₂(ω) where q = n, and (iii) S₃(ω) where q > n, and then summing them up as:

\[ S_N(\omega) = S_{N1}(\omega) + S_{N2}(\omega) + S_{NS}(\omega) \]

Detailed derivations of S₁(ω), S₂(ω) and S₃(ω) are given in the Appendix (Section 8.1). Substituting eqns. 30, 32 and 35 into eqn. 10 will result in:

\[ S_N(\omega) = \frac{V}{j\omega} \left[ 5 - 4 \sin^2 \left( \frac{\alpha T_s}{4} \right) \right] + \left[ \frac{2}{4} - 4 \sin^2 \left( \frac{\alpha T_s}{4} \right) \right] \]

\[ \times \text{Re} \left\{ \frac{G}{(1-G)^2} \left[ N(1 - G) - (1 - G^N) \right] \right\} \]

with G given in eqn. 28.
The power spectral density of the truncated signal can be obtained by substituting eqns. 7 and 8 into eqn. 6 to produce:

$$P(\omega) = \frac{4V^2 \sin^2\left(\frac{\omega T_s}{4}\right)}{\omega^2 T_s \left(1 + \alpha + \frac{2M-1}{2}\right)} \cdot \lim_{N \to \infty} \left\{ \frac{S_N(\omega)}{N} \right\}$$

(12)

To simplify eqn. 12, we need to investigate the possible values of $S_N(\omega)$ as given in eqn. 11, which largely depend on $G$, where $|G| \leq 1$ as shown in the Appendix (Section 8.2). Therefore, only two cases needed to be investigated as outlined below.

$\textbf{Case 1: where } |G| < 1$

From eqns. 36 and 37, $|G| < 1$ when $e^{j\omega T_s} \neq 1$, i.e. $\omega \neq 2\pi K T_s$ where $K$ is a positive integer.

Here, $\lim_{N \to \infty} S_N(\omega) = 0$ and therefore, from eqn. 11,

$$\lim_{N \to \infty} \left\{ \frac{S_N(\omega)}{N} \right\} = \frac{1}{2} \left[ 5 - 4 \sin^2\left(\frac{\alpha \omega T_s}{4}\right) \right]$$

$$+ \left[ 9 - 8 \sin^2\left(\frac{\omega T_s}{4}\right) \right] Re \left[ \frac{G}{1-G} \right]$$

(13)

Substituting eqn. 13 into eqn. 12 results in:

$$2V^2 \sin^2\left(\frac{\alpha \omega T_s}{4}\right) \left\{ 5 - 4 \sin^2\left(\frac{\alpha \omega T_s}{4}\right) \right\}$$

$$P(\omega) = \frac{1}{\omega^2 T_s \left(1 + \alpha + \frac{2M-1}{2}\right)} \left[ 9 - 8 \sin^2\left(\frac{\omega T_s}{4}\right) \right] Re \left[ \frac{G}{1-G} \right]$$

(14)

The expression in eqn. 14 gives the PSD profile of the DH-PIM signal when it is finite. (Here, $|G| \neq 1$ and $\omega \neq 0$).

$\textbf{Case 2: where } G = 1$

From eqn. 36, $G = 1$ when $e^{j\omega T_s} = 1$, i.e. $\omega = 2\pi K T_s$ where $K$ is a positive integer.

Here, the expression in eqn. 11 is indeterminate, but applying L'Hôpital's rule [14] with $G \to 1$ gives:

$$\lim_{G \to 1} \left[ \frac{N(1-G)-(1-G^N)}{(1-G)^2} \right] = \frac{N(N-1)}{2}$$

Thus from eqn. 11

$$S_N(\omega) = \frac{N}{4} \left[ 10 - 8 \sin^2\left(\frac{\alpha \omega T_s}{4}\right) \right]$$

$$+ \left( N-1 \right) \left[ 9 - 8 \sin^2\left(\frac{\omega T_s}{4}\right) \right]$$

(15)

Substituting eqn. 15 into eqn. 12 results in:

$$P(\omega) = \frac{\sin^2\left(\frac{\omega T_s}{4}\right)}{\omega^2 T_s \left(1 + \alpha + \frac{2M-1}{2}\right)} \left\{ 10 - 8 \sin^2\left(\frac{\alpha \omega T_s}{4}\right) \right\}$$

$$+ \left( N-1 \right) \left[ 9 - 8 \sin^2\left(\frac{\omega T_s}{4}\right) \right]$$

(16)

Depending on the value of $K$, eqn. 16 will tend to 0 or $\infty$, as discussed below:

(i) For $K = 0$, $\omega = 0$, and applying L'Hôpital's rule (since at $\omega = 0$ eqn. 16 is indeterminate). eqn. 16 will tend towards infinity as:

$$P(0) = \frac{V^2}{T_s} \lim_{N \to \infty} \left\{ \frac{9N + 1}{\left(1 + \alpha + \frac{2M-1}{2}\right)} \right\}$$

$$P(0) = \frac{\sin^2\left(\frac{\omega T_s}{4}\right)}{\omega^2} \right\}$$

(17)

(ii) For $K = 0$, $\omega = 0$, and applying L'Hôpital's rule (since at $\omega = 0$ eqn. 16 is indeterminate). eqn. 16 will tend towards infinity as:

$$P(0) = \frac{\alpha^2 V^2 T_s}{16} \lim_{N \to \infty} \left\{ \frac{9N + 1}{1 + \alpha + \frac{2M-1}{2}} \right\}$$

(18)

(iii) For all other frequencies of the form $\omega = 2\pi K T_s$, $P(\omega)$ tends to zero as $K \to \infty$.

4 Results and discussion

Using eqn. 19, the calculated PSD of DH-PIM signal is plotted for $M = 4$ bits, and different values of $\alpha$ for $\alpha = 1$ the spectral profile contains a DC component, a distinct slot component and its harmonics, and nulls at even multiples of the slot frequency, see Fig. 3, which also includes corresponding simulation result. For $\alpha = 2$ the nulls coincide with the slot component and its harmonics, thus suppressing them to zero, see Fig. 4. For $\alpha > 2$ the picture is broadly similar. As $\alpha$ increases the slot components remain

![Fig. 3 Predicted and simulated power spectral density of DH-PIM for M = 4 and \alpha = 1 against normalised frequency](image-url)
fixed in the frequency spectrum but the nulls shift towards zero. Also, for all even values of \( \alpha \) the slot components are masked by the nulls and therefore, at the receiver end a simple phase locked loop (PLL) circuit cannot be used to extract the slot component for synchronisation purpose. However, the slot frequency component can be extracted by employing a nonlinear device followed by a PLL circuit [15, 16]. The above results confirm that the spectrum consists of distinct frequency components at the slot frequency and its harmonics, other than those at which the pulse transform gives zeros as given by eqn. 16. This confirms that the presence of the slot components and the locations of nulls are affected by the pulse shape.

![Fig. 4: Predicted and simulated power spectral density of DH-PIP for \( M = 4 \) and \( \alpha = 2 \) against normalised frequency](image)

To confirm the predicted result, the complete DH-PIP system was simulated using MATLAB. A random data generator was used to obtain 8000 consecutive random DH-PIP symbols within a specific period at specified bit resolution. A bit resolution \( M = 4 \) was initially chosen to give 16 different symbols with shortest and longest symbol length of 3 \( T_s \) and 10 \( T_s \), respectively (for \( \alpha = 2 \)). The probability of an occurrence of a particular symbol was held at 1/16. The PSD was obtained and the results are shown in Figs. 3 and 4 for different values of \( \alpha \). As predicted, the spectral profile contains DC, nulls (or low power slot frequency components) and, under the conditions specified above, distinctive frequency components at the odd harmonics of the slot frequency, thus confirming the validity of the theoretical model.

The DH-PIP PSD contains a non-zero DC component which is marginally higher than those of PPM and PIM at higher bit resolution, see Fig. 5. From eqn. 18 the DC component of DH-PIP normalised to \( M = 4 \) and \( \alpha = 1 \) is given by:

\[
P_{\text{nor}}(0) = \frac{11\alpha^2}{1 + 2\alpha + 2^{M-1}}
\]  

Fig. 5 shows the predicted normalised DC component of DH-PIP against \( M \) for different values of \( \alpha \). The DC component depends on \( M \) and \( \alpha \), decreasing to lower values with increasing \( M \). For \( \alpha = 1 \) and \( M > 6 \) bits, a DC level less than 0.2 is achievable, a characteristic which is desirable in an application such as optical wireless communication systems. Fig. 5 also shows the simulated normalised DC component for the same parameters using 8000 consecutive random DH-PIP data, which closely agrees with the predicted results. Also, the normalised simulated DC components of PPM and PIM are shown for \( M = 4 \) and a pulse duty cycle of 100% in both cases.

![Fig. 5: DC components of DH-PIP, PIM and PPM, normalised to DC component of DH-PIP when \( M = 4 \) and \( \alpha = 1 \), against bit resolution (\( M \))](image)

The position and amplitude of the distinct slot frequency component changes as the pulse duty cycle is varied. For \( \alpha = 2 \), nulls frequency component coincides with the slot frequency components, and the spectral profile resembles a general sinc envelope shape, cancelling out the slot components, see Fig. 4. The presence of the slot component in the spectrum suggests that a PLL can be used at the receiver to extract the slot frequency component directly from the incoming DH-PIP data stream for slot synchronisation similar to that in PPM and PIM. The amplitude of the slot component varies as a function of \( M \) and \( \alpha \). From eqn. 16, the peak amplitude of the fundamental slot component normalised to \( M = 4 \) bits and \( \alpha = 1 \) is given by:

\[
P_{\text{nor,slot}} = \frac{11}{1 + 2\alpha + 2^{M-1}}
\]  

Fig. 6 shows the predicted and simulated normalised peak amplitude of the fundamental slot component versus the bit resolution for two different values of \( \alpha \). At lower bit resolution the strength of the slot component depends more strongly on \( \alpha \), decreasing with higher values of \( \alpha \). However, at higher bit resolution the slot components become less distinct compared to the lower bit resolution and less dependent on \( \alpha \) as shown in Fig. 6. Fig. 7 presents...
a three-dimensional view of, simulation results, for $\alpha = 1$, incorporating essential features of Figs. 3 and 6 as already discussed.

5 Conclusions

A mathematical model has been developed to represent the DH-PIM pulse train and power spectral density and it is shown that it is capable of predicting the spectral profile. Results obtained indicate that there exist distinct spectral components at odd harmonics of the slot frequency, which can be used for slot synchronisation similar to that in PPM and PIM. It is also shown that, similar to PPM and PIM, DH-PIM contains a non-zero DC component which is marginally higher than those of PPM and PIM at $M = 6$ bits. Simulation results presented are in close agreement with the theoretical predictions, thus validating the DH-PIM spectral model.

6 Acknowledgment

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7 References

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8 Appendix

8.1 Derivation of $S_{N1}(\omega)$, $S_{N2}(\omega)$, and $S_{N3}(\omega)$

Here we set out to find the expressions for $S_{N1}(\omega)$, $S_{N2}(\omega)$, and $S_{N3}(\omega)$ as outlined in eqn. 10.

(i) Taking $q < n$ in eqn. 9 gives:

$$S_{N1}(\omega) = \sum_{n=1}^{N-1} \sum_{q=0}^{n-1} E \left[ (1 + \alpha e^{-j\omega T_s/2}) (1 + \beta e^{j\omega T_s/2}) e^{-j\omega T_s (\alpha+1) (n-q)} \sum_{k=q}^{n-1} d_k \right]$$

(ii) Taking $q = n$ in eqn. 9 gives:

$$S_{N2}(\omega) = \sum_{n=1}^{N-1} \sum_{q=0}^{n} E \left[ (1 + \alpha \beta e^{-j\omega T_s/2} + \beta e^{j\omega T_s/2}) e^{-j\omega T_s (\alpha+1) (n-q)} \sum_{k=q}^{n-1} d_k \right]$$

(iii) Taking $q > n$ in eqn. 9 gives:

$$S_{N3}(\omega) = \sum_{n=1}^{N-1} \sum_{q=n+1}^{N} E \left[ (1 + \alpha e^{-j\omega T_s/2} + \beta e^{j\omega T_s/2}) e^{-j\omega T_s (\alpha+1) (n-q)} \sum_{k=q}^{n-1} d_k \right]$$

\( S_{N1}(\omega) \)
\[
= \sum_{n=1}^{N-1} \sum_{q=0}^{n-1} \left\{ E(1 + h_n h_q + h_n e^{-j\omega T_s/2} + h_q e^{j\omega T_s/2}) \right\} \\
\times \left[ e^{-j\omega T_s \sum_{k=q}^{n-1} d_k} \right] e^{-j\omega T_s (\alpha + 1)(n-q)}
\]
\[
E[h_n h_q] = \frac{1}{2}
\]
Furthermore, \( h_n h_q \in \{0,0.1\} \) and so the expected value of \( h_n h_q \) is
\[
E[h_n h_q] = \frac{1}{4}
\]
Consequently, the first factor reduces to:
\[
E\left[ (1 + h_n h_q + h_n e^{-j\omega T_s/2} + h_q e^{j\omega T_s/2}) \right] = \frac{5}{4} + \cos\left( \frac{\alpha T_s}{2} \right) = \frac{9}{4} - 2\sin^2\left( \frac{\alpha T_s}{4} \right)
\]
and the second factor to:
\[
E\left( e^{-j\omega T_s \sum_{k=q}^{n-1} d_k} \right) = E\left[ \prod_{k=q}^{n-1} e^{-j\omega T_s d_k} \right]
\]
\[
= \prod_{k=q}^{n-1} \left( \frac{1}{2^{M-1}} \sum_{d=0}^{2^{M-1}-1} e^{-j\omega T_s d} \right)
\]
\[
= \prod_{k=q}^{n-1} \left[ \frac{1}{2^{M-1}} \left( \frac{1 - e^{-j\omega T_s 2^{M-1}}}{1 - e^{-j\omega T_s}} \right) \right]^{n-q}
\]
Thus, substituting eqns. 25 and 26 in eqn. 22 gives:
\[
S_{N1}(\omega) = \sum_{n=1}^{N-1} \sum_{q=0}^{n-1} \left\{ \frac{1}{4} - 2\sin^2\left( \frac{\alpha T_s}{2} \right) \right\} \left( \frac{1}{2^{M-1}} \cdot \frac{1 - e^{-j\omega T_s 2^{M-1}}}{1 - e^{-j\omega T_s}} \right)^{n-q} e^{-j\omega T_s (n-q)(\alpha+1)}
\]
\[
S_{N1}(\omega) = \frac{9}{4} - 2\sin^2\left( \frac{\alpha T_s}{4} \right) \sum_{n=1}^{N-1} \sum_{q=0}^{n-1} \left( \frac{1 - e^{-j\omega T_s 2^{M-1}}}{1 - e^{-j\omega T_s}} \right)^{n-q} e^{-j\omega T_s (n-q)(\alpha+1)}
\]
Now letting
\[
G = \left( \frac{1 - e^{-j\omega T_s 2^{M-1}}}{1 - e^{-j\omega T_s}} \cdot \frac{e^{-j\omega T_s (\alpha+1)}}{2^{M-1}} \right)
\]
and using the result
\[
\sum_{n=1}^{N-1} \sum_{q=0}^{n-1} G^{n-q} = \frac{G}{(1-G)^2} \left[ N(1-G) - (1-G^N) \right]
\]
in eqn. 27 gives
\[
S_{N1}(\omega) = \left[ \frac{9}{4} - 2\sin^2\left( \frac{\alpha T_s}{4} \right) \right] \times \frac{G}{(1-G)^2} \left[ N(1-G) - (1-G^N) \right]
\]
(ii) Taking \( q = n \) in eqn. 9 gives
\[
S_{N2}(\omega) = \sum_{n=0}^{N-1} E\left[ (1 + h_n e^{-j\omega T_s/2})(1 + h_n e^{j\omega T_s/2}) \right]
\]
\[
= \sum_{n=0}^{N-1} E\left[ (1 + h_n^2 + h_n e^{-j\omega T_s/2} + h_n e^{j\omega T_s/2}) \right]
\]
\[
h_n^2 \in \{0,1\}, \text{ thus}
\]
\[
E[h_n^2] = \frac{1}{2}
\]
Hence, \( S_{N2}(\omega) \) can be written as
\[
S_{N2}(\omega) = \sum_{n=0}^{N-1} \left[ \frac{3}{2} + \cos\left( \frac{\alpha T_s}{2} \right) \right] = \frac{N}{2} \left[ 5 - 4\sin^2\left( \frac{\alpha T_s}{4} \right) \right]
\]
(iii) Taking \( q > n \) in eqn. 9 gives
\[
S_{N3}(\omega) = \sum_{n=0}^{N-2} \sum_{q=n+1}^{N-1} E\left[ (1 + h_n e^{-j\omega T_s/2})(1 + h_n e^{j\omega T_s/2}) \times e^{-j\omega T_s (\alpha+1)(n-q)} e^{j\omega T_s \sum_{k=n}^{n+1} d_k} \right]
\]
which can be rewritten as:
\[
S_{N3}(\omega) = \sum_{n=0}^{N-1} \sum_{q=1}^{n} E\left[ (1 + h_n e^{-j\omega T_s/2})(1 + h_n e^{j\omega T_s/2}) \times e^{-j\omega T_s (\alpha+1)(n-q)} e^{j\omega T_s \sum_{k=n}^{n+1} d_k} \right]
\]
Interchanging \( q \) and \( n \) gives:
\[
S_{N3}(\omega) = \sum_{n=0}^{N-1} \sum_{q=0}^{n} E\left[ (1 + h_n e^{-j\omega T_s/2})(1 + h_n e^{j\omega T_s/2}) \times e^{j\omega T_s (\alpha+1)(n-q)} e^{j\omega T_s \sum_{k=n}^{n+1} d_k} \right]
\]
Therefore,
\[ S_{N_3}(\omega) = S_{N_1}(\omega) \]  \hspace{1cm} (34)

and
\[ S_{N_1}(\omega) + S_{N_3}(\omega) = 2\text{Re}[S_{N_1}(\omega)] \]  \hspace{1cm} (35)

8.2 Proof that \(|G| \leq 1\)

Eqn. 28 can be rewritten as:
\[
G = \frac{1}{2^{M-1}} \left\{ 1 + e^{-j\omega T_s} + e^{-j2\omega T_s} + \ldots + e^{-j(2^{M-1}-1)\omega T_s} \right\} \cdot e^{-j\omega(\alpha+1)T_s}
\]

Therefore, the absolute value of this is given as:
\[ |G| = \frac{1}{2^{M-1}} \left| 1 + e^{-j\omega T_s} + e^{-j2\omega T_s} + \ldots + e^{-j(2^{M-1}-1)\omega T_s} \right| \]  \hspace{1cm} (36)

Hence,
\[
|G| \leq \frac{1}{2^{M-1}} \left\{ 1 + |e^{-j\omega T_s}| + |e^{-j2\omega T_s}| + \ldots + |e^{-j(2^{M-1}-1)\omega T_s}| \right\}
\]

and thus,
\[ |G| \leq 1 \]  \hspace{1cm} (37)